Practice Exam 2 — Functional Analysis (WIFA–08)

University of Groningen

Instructions

1. The use of calculators, books, or notes is not allowed.
2. All answers need to be accompanied with an explanation or a calculation: only answering "yes", "no", or "42" is not sufficient.

Problem 1

Let X and Y be Banach spaces. Prove that:

(a) $\|(x, y)\|_\infty = \max\{\|x\|, \|y\|\}$ is a norm on the Cartesian product $X \times Y$;

(b) $(X \times Y, \|\cdot\|_\infty)$ is a Banach space (i.e., every Cauchy sequence is convergent);

(c) $(X \times Y, \|\cdot\|_\infty)$ is *not* a Hilbert space.

Problem 2

Let X be Hilbert space with an orthonormal basis $\{e_n \,:\, n \in \mathbb{N}\}$. Let (λ_n) be a bounded sequence in $\mathbb{K}$ and consider the following linear operator:

$$T : X \to X, \quad Tx = \sum_{n=1}^{\infty} \lambda_n (x, e_n) e_n, \quad x \in X.$$

Prove the following statements:

(a) $\|T\| = \sup_{n\in\mathbb{N}} |\lambda_n|$;

(b) $\lambda_n \to 0$ implies that T is compact;

(c) $\sigma(T) = \operatorname{clos}\{\lambda_n \,:\, n \in \mathbb{N}\}$.

Problem 3

(a) Formulate the uniform boundedness principle.

(b) Let X be a Hilbert space, let $T : X \to X$ be a linear operator, and assume that

$$(Tx, y) = (x, Ty) \quad \text{for all} \quad x, y \in X.$$

Prove the following statements:

(i) for $y \neq 0$ the map $f_y : X \to \mathbb{K}$ defined by $f_y(x) = (Tx, y)/\|y\|$ is linear;

(ii) $\sup_{y\neq 0} |f_y(x)| < \infty$ for all $x \in X$;

(iii) T is bounded.

Problem 4

Let X be a normed linear space. For $V \subset X$ and $Z \subset X'$ nonempty subsets, define

$$V^{\perp} = \{f \in X' : \ f(x) = 0 \text{ for all } x \in V\},$$

$$^{\perp}Z = \{x \in X : \ f(x) = 0 \text{ for all } f \in Z\}.$$

Prove the following statements:

(a) $^{\perp}Z$ is a linear subspace of X;

(b) $^{\perp}Z$ is closed in X;

(c) $Z_1 \subset Z_2 \subset X' \Rightarrow {}^{\perp}Z_2 \subset {}^{\perp}Z_1$;

(d) $Z \subset ({}^{\perp}Z)^{\perp}$.

End of test (90 points)

Solution of Problem 1

(a) Clearly $\|(x,y)\|_\infty \geq 0$. If $\|(x,y)\|_\infty = 0$ then both $\|x\| = 0$ and $\|y\| = 0$ which shows that $(x,y) = (0,0)$.

If $\lambda \in \mathbb{K}$ then

$$\begin{aligned}\|\lambda(x,y)\|_\infty &= \|(\lambda x, \lambda y)\|_\infty \\ &= \max\{\|\lambda x\|, \|\lambda y\|\} \\ &= \max\{|\lambda|\|x\|, |\lambda|\|y\|\} \\ &= |\lambda| \max\{\|x\|, \|y\|\} \\ &= |\lambda|\|(x,y)\|_\infty.\end{aligned}$$

Finally,

$$\begin{aligned}\|(x,y) + (u,v)\|_\infty &= \|(x+u, y+v)\|_\infty \\ &= \max\{\|x+u\|, \|y+v\|\} \\ &\leq \max\{\|x\| + \|u\|, \|y\| + \|v\|\} \\ &\leq \max\{\|x\|, \|y\|\} + \max\{\|u\|, \|v\|\} \\ &= \|(x,y)\|_\infty + \|(u,v)\|_\infty.\end{aligned}$$

(b) Let (x_n, y_n) be a Cauchy sequence in $X \times Y$. For each $\varepsilon > 0$ there exists $N > 0$ such that

$$\begin{aligned} n,m \geq N \quad &\Rightarrow \quad \|(x_n, y_n) - (x_m, y_m)\|_\infty \leq \varepsilon \\ &\Rightarrow \quad \|(x_n - x_m, y_n - y_m)\|_\infty \leq \varepsilon \\ &\Rightarrow \quad \|x_n - x_m\| \leq \varepsilon \quad \text{and} \quad \|y_n - y_m\| \leq \varepsilon.\end{aligned}$$

This means that (x_n) is a Cauchy sequence in X and (y_n) is a Cauchy sequence in Y. Since X and Y are assumed to be Banach spaces there exist $x \in X$ and $y \in Y$ such that

$$\|x_n - x\| \to 0 \quad \text{and} \quad \|y_n - y\| \to 0.$$

This implies that

$$\begin{aligned}\|(x_n, y_n) - (x,y)\|_\infty &= \|(x_n - x, y_n - y)\|_\infty \\ &= \max\{\|x_n - x\|, \|y_n - y\|\} \to 0.\end{aligned}$$

We conclude that (x_n, y_n) has the limit (x,y).

(c) Pick $x \in X$ with $\|x\| = 1$ and $v \in Y$ with $\|v\| = 1$, then

$$\|(x,0)\|_\infty = 1, \quad \|(0,v)\|_\infty = 1, \quad \|(x,v)\|_\infty = 1, \quad \|(x,-v)\|_\infty = 1,$$

which shows that

$$\|(x,0) + (0,v)\|_\infty^2 + \|(x,0) - (0,v)\|_\infty^2 \neq 2\|(x,0)\|_\infty^2 + 2\|(0,v)\|_\infty^2.$$

Since the parallelogram identity does not hold, it follows that the norm $\|\cdot\|_\infty$ on $X \times Y$ is not induced by an inner product. Hence, $(X \times Y, \|\cdot\|_\infty)$ is not a Hilbert space.

Solution of Problem 2

(a) For each $x \in X$ we have

$$\|Tx\|^2 = \sum_{n=1}^{\infty} |\lambda_n|^2 \, |(x, e_n)|^2 \leq \sup_{n\in\mathbb{N}} |\lambda_n|^2 \sum_{n=1}^{\infty} |(x, e_n)|^2 = \sup_{n\in\mathbb{N}} |\lambda_n|^2 \|x\|^2,$$

which implies that

$$\|T\| = \sup_{x\neq 0} \frac{\|Tx\|}{\|x\|} \leq \sup_{n\in\mathbb{N}} |\lambda_n|.$$

Now write $K = \sup_{n\in\mathbb{N}} |\lambda_n|$. For each $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that $|\lambda_N| > K - \varepsilon$. Hence,

$$\frac{\|Te_N\|}{\|e_N\|} = |\lambda_N| > K - \varepsilon.$$

Hence we conclude

$$\|T\| = \sup_{x\neq 0} \frac{\|Tx\|}{\|x\|} = \sup_{n\in\mathbb{N}} |\lambda_n|.$$

(b) Assume that $\lambda_n \to 0$. Define the operator

$$T_k : X \to X, \quad T_k x = \sum_{n=1}^{k} \lambda_n (x, e_n) e_n, \quad x \in X.$$

By part (a) it follows that $\|T_k\| = \max\{|\lambda_1|, \ldots, |\lambda_k|\}$ so that T_k is bounded. In addition, $\dim \operatorname{ran} T_k \leq k < \infty$. Hence, T_k is compact for each $k \in \mathbb{N}$. Also note that

$$\|T - T_k\| = \sup_{n>k} |\lambda_n|.$$

Let $\varepsilon > 0$ be arbitrary. There exists $N > 0$ such that

$$\begin{aligned} n \geq N \quad &\Rightarrow \quad |\lambda_n| \leq \varepsilon \\ &\Rightarrow \quad \|T - T_n\| \leq \varepsilon. \end{aligned}$$

This implies that T is compact as well.

(c) Clearly, each λ_n is an eigenvalue of T with corresponding eigenvector e_n. Hence, $\{\lambda_n : n \in \mathbb{N}\} \subset \sigma(T)$. Since the spectrum of a linear operator is closed it also follows that $\operatorname{clos}\{\lambda_n : n \in \mathbb{N}\} \subset \sigma(T)$.

If $\lambda \notin \operatorname{clos}\{\lambda_n : n \in \mathbb{N}\}$ then there exists a $\delta > 0$ such that $|\lambda - \lambda_n| > \delta$ for each $n \in \mathbb{N}$. Note that

$$(T - \lambda)^{-1} x = \sum_{n=1}^{\infty} \frac{1}{\lambda_n - \lambda} (x, e_n) e_n, \quad x \in X.$$

Hence,

$$\|(T-\lambda)^{-1}x\|^2 = \sum_{n=1}^{\infty} \frac{|(x,e_n)|^2}{|\lambda_n - \lambda|^2} \le \frac{1}{\delta^2} \sum_{n=1}^{\infty} |(x,e_n)|^2 = \frac{1}{\delta^2}\|x\|^2.$$

Hence, $(T-\lambda)^{-1}$ is bounded so that $\lambda \in \rho(T)$. We conclude that $\sigma(T) = \operatorname{clos}\{\lambda_n : n \in \mathbb{N}\}$.

Solution of Problem 3

(a) Let X be a Banach space and let Y be a normed linear space. Let $F \subset B(X,Y)$ and assume that

$$M = \{x \in X : \sup_{T\in F} \|Tx\| < \infty\}$$

is nonmeager. Then the elements $T \in F$ are uniformly bounded:

$$\sup_{T\in F} \|T\| < \infty.$$

(b) (i) The fact that $f_y : X \to \mathbb{K}$ defined by $f_y(x) = (Tx,y)/\|y\|$ is a linear map follows from:

$$\begin{aligned} f_y(\lambda x + \mu z) &= \frac{(T(\lambda x + \mu z), y)}{\|y\|} \\ &= \frac{(\lambda Tx + \mu Tz, y)}{\|y\|} \\ &= \frac{\lambda (Tx,y)}{\|y\|} + \frac{\mu (Tz,y)}{\|y\|} \\ &= \lambda f_y(x) + \mu f_y(z). \end{aligned}$$

(ii) Let $x \in X$ be arbitrary, then

$$|f_y(x)| = \frac{|(Tx,y)|}{\|y\|} \le \frac{\|Tx\|\,\|y\|}{\|y\|} = \|Tx\|.$$

This shows that

$$\sup_{y\neq 0} |f_y(x)| < \infty$$

for all $x \in X$.

(iii) By the uniform boundedness principle it follows that $\sup_{y\neq 0}\|f_y\| < \infty$. Since $(Tx,y) = (x,Ty)$ it follows with $x = Ty/\|y\|$ that

$$\frac{\|Ty\|^2}{\|y\|^2} = f_y\left(\frac{Ty}{\|y\|}\right) \le \|f_y\| \frac{\|Ty\|}{\|y\|}$$

so that

$$\|T\| = \sup_{y\neq 0} \frac{\|Ty\|}{\|y\|} \le \sup_{y\neq 0} \|f_y\| < \infty$$

which shows that T is bounded.

Solution of Problem 4

(a) Assume that $x, y \in {}^{\perp}Z$ and $\lambda, \mu \in \mathbb{K}$. If $f \in Z$, then

$$f(\lambda x + \mu y) = \lambda f(x) + \mu f(y) = 0,$$

which implies that $\lambda x + \mu y \in {}^{\perp}Z$ as well. This shows that ${}^{\perp}Z$ is a linear subspace of X.

(b) Assume that $x_n \in {}^{\perp}Z$ for all $n \in \mathbb{N}$ and $x_n \to x$. If $f \in Z$, then $f(x_n) = 0$ for all $n \in \mathbb{N}$ so that

$$\begin{aligned} |f(x)| &= |f(x - x_n) + f(x_n)| \\ &\leq |f(x - x_n)| + |f(x_n)| \\ &= |f(x - x_n)| \\ &\leq \|f\| \, \|x_n - x\| \to 0. \end{aligned}$$

This implies that $f(x) = 0$ so that $x \in {}^{\perp}Z$ as well. Hence, ${}^{\perp}Z$ is closed in X.

(c) Let $x \in {}^{\perp}Z_2$, then $f(x) = 0$ for all $f \in Z_2$. Since $Z_1 \subset Z_2$ it follows that $f(x) = 0$ for all $f \in Z_1$, which means that $x \in {}^{\perp}Z_1$. Hence, ${}^{\perp}Z_2 \subset {}^{\perp}Z_1$.

(d) If $f \in Z$, then $f(x) = 0$ for all $x \in {}^{\perp}Z$. Hence, $f \in ({}^{\perp}Z)^{\perp}$.